

Versatile Learning of Mathematics

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It was many years ago now, in 1971, that as a young undergraduate in mathematics at Warwick University, I was first introduced to Richard Skemp's ideas by a young lecturer named David Tall. I read the paperback *The Psychology of Learning Mathematics* with great interest. It was not until I was a graduate student of mathematics education, in 1983, (once again working with David) that I re-read some of Richard's work, and yet, as a teacher of mathematics in schools it had remained with me. I read *Intelligence, Learning and Action* and was struck by the power of the ideas expressed in it and the simplicity of the language in which they were presented, especially in comparison with other texts I read. I learned the valuable lesson that powerful ideas can be communicated in simple terms and do not require a facade of convoluted definitions and expressions in their presentation.

Another educationist, Ausubel (1968, p. iv), well-known for his concept of meaningful learning, said, "If I had to reduce all of educational psychology to just one principle I would say this: The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly." Similarly, today, many constructivist mathematics educators would maintain as a central tenet that the mathematics children know should be the basis on which to teach mathematics (e.g. Steffe, 1991). Skemp, agreed, remarking that "our conceptual structures are a major factor of our progress" (1979c, p. 113). Since our existing schemas serve either to promote or restrict the association of new concepts, then the quality of what an individual already knows is a primary factor affecting their ability to understand. The existence of a wide level of agreement on this point indicates very strongly that it is something which mathematics educators should take to heart. However, one does not have to have been a teacher for very long before being faced with a dilemma. Many of us will have experienced the reaction of students when, after we have spent some considerable time trying to develop the ideas and concepts of differentiation, we introduce the rule for antidifferentiation

of x^n . They may say “well why didn’t you just tell us that was how to do it?” Too often it seems, the students’ focus is on how ‘to do’ mathematics rather than on what mathematics is about, what its objects and concepts are. This procedural view of mathematics is, sadly, often reinforced by teachers who succumb to the pressure and only tell students how to do the mathematics.

I am particularly interested in the way in which our conceptual structures enable us to relate the procedural/process aspects of mathematics with the conceptual ideas, such as why the formula is correct. The essentially sequential nature of algorithms often contrast with the more global or holistic nature of conceptual thinking, and, the ways in which we, as individuals, construct schemas which enable us to relate the two in a *versatile* way is of great importance. The full meaning of the term ‘*versatile*’, as used here will emerge during this discussion but, suffice it to say at this stage, that it will be used in a way which will have the essence of the usual English meaning, but, will take on a more precise technical sense, which will be explained during a discussion of the nature of our conceptual structures.

Having acknowledged the importance of our mental schemas in building mathematical understanding, some questions worth considering in any attempt to encourage *versatile* learning of mathematics include:

- How are schemas constructed?
- How can we identify and describe the quality of our constructions?
- How do our schemas influence our perception of the objects and procedures in mathematics, and their relationship?
- What experiences will help us improve the quality of our constructions so that we can build a *versatile* view of mathematics?

Skemp’s theory has much to offer towards building answers to these sorts of questions, and in the rest of this paper I will seek to show how valuable it is. First we will consider his model of intelligence and its implications for improving the quality of learning and understanding.

Skemp's Model of Intelligence

The basis of Skemp's theory of learning (Skemp 1979c, p. 89) is a model which describes intelligence as an activity in which learning is "a goal-directed change of state of a director system towards states which, for the assumed environment, make possible optimal functioning." According to this model of intelligence, we all engage in mental construction of reality by building and testing a schematic knowledge structure (Skemp, 1985). For Skemp (1979c, p. 219), a schema is "a conceptual structure existing in its own right, independently of action.", and he describes (*ibid*, p. 163) three modes which each of us may use to build and test such structures:

Reality building: from our own encounters with actuality; from the realities of others; from within.

Reality testing: against expectation of events in actuality; comparison with the realities of others; comparison with one's own existing knowledge and beliefs.

This process of mental construction involves two director systems, which Skemp describes as delta-one (Δ_1) and delta-two (Δ_2). The former is a kind of sensori-motor system which "receives information ... compares this with a goal state, and with the help of a plan which it constructs from available schemas, takes the operand from its present state to its goal state." (Skemp, 1979b, p. 44). Delta-two on the other hand is a goal directed mental activity, whose operands are in delta-one, and its job is to optimise the functioning of delta-one (Skemp, 1979a). Hence the construction of concepts in a schema, or knowledge structure, may be by abstraction via direct sensory experience from actuality (primary concepts) using delta-one, or by derivation from other concepts (secondary concepts), using delta-two. In turn, the acquisition of new concepts may require expansion or re-construction of the relevant schema, altering it to take account of a concept for which it is relevant but not adequate (Skemp, 1979c, p. 126). Factors such as the frequency of contributory experiences, the existence of *noise* (irrelevant input) and the availability of lower order concepts, may affect one's ability to form concepts using the director systems.

Skemp (1979b, p. 48) outlines two modes of mental or 'intelligent' activity which take place in the context of delta-one and delta-two, namely intuitive and reflective:

In the intuitive mode of mental activity, consciousness is centred in delta-one. In the reflective mode, consciousness is centred in delta-two. 'Intuitive' thus refers to spontaneous processes, those within delta-one, in which delta-two takes part either not at all, or not consciously. 'Reflective' refers to conscious activity by delta-two on delta-one.

The concept of *reflective intelligence* was one of the earliest which Skemp illuminated (see Skemp, 1961, 1978), and while acknowledging that the term is Piaget's, Skemp views his model of the concept differently, saying "When the concept of reflective intelligence was first introduced...I acknowledged that this term was borrowed from Piaget, and noted also that my model was not the same as his." (Skemp, 1979c, p. 218). For Skemp *reflective intelligence* involves the awareness of our own concepts and schemas, examining and improving them, thereby increasing our ability to understand. The methods available for doing so include physical and mental experimentation, generalisation and systematising knowledge, by looking for conceptual connections. It is this *conscious*, reflective mental activity which Skemp (1961) considered vital for successful building of mathematical knowledge structures, and it is this which he believes increases mathematical performance. He (Skemp, 1961, p. 49) illustrates this with reference to algebra, commenting:

The transition to algebra, however, involves deliberate generalisation of the concepts and operations of arithmetic...Such a process of generalisation does require awareness of the concepts and operations themselves. Since these are not physical objects, perceivable by the external senses, but are mental, this transition requires the activity of reflective thought. And further, the generalisation requires not only awareness of the concepts and operations but perceptions of their inter-relations. This involves true reflective intelligence.

Thus, for Skemp, the change in the quality of thinking involved in the transition from arithmetic to algebra is brought about by 'awareness of' and 'deliberate generalisation of' the concepts and operations of arithmetic; that is by reflective intelligence.

The Qualitative Nature of Mathematical Understanding

Having briefly discussed Skemp's view of the construction of mental schemas, we now ask, 'what about the qualitative nature of these schemas?' One of the first of Skemp's ideas I encountered, and which has stayed with me, was a strikingly simple, but extremely useful, definition of understanding, a term which is often used, but less

commonly defined or explained. According to Skemp (1979c, p. 148), “to understand a concept, group of concepts, or symbols is to connect it with an appropriate schema.” Skemp further clarified what he saw as understanding by describing two types, now well known in mathematics education, as instrumental and relational. This appreciation of the qualitative nature of understanding is no doubt one of the most valuable insights provided by Skemp’s model of intelligence. In his landmark paper (Skemp, 1976) he described instrumental understanding as learning ‘how to’, involving learning by rote, memorising facts and rules. In contrast, relational learning, or learning ‘why to’, consists primarily of relating a task to an *appropriate* schema. Whilst this has received wide acceptance as a valuable insight, there were attempts to extend and re-shape some of the ideas he presented, and following this Skemp (1979b) created another category of understanding, akin to formal understanding, thus increasing them to three; instrumental, relational and logical. This last type he describes (Skemp, 1979b, p. 47) in these terms:

Logical understanding is evidenced by the ability to demonstrate that what has been stated follows of logical necessity, by a chain of inferences, from (i) the given premises, together with (ii) suitably chosen items from what is accepted as established mathematical knowledge (axioms and theorems).

Hence the acquiring of logical understanding implies that the individual not only has relational understanding but is able to demonstrate evidence of such understanding to others by means of “a valid sequence of logical inferences” (Skemp, 1979a, p. 200).

The qualitative differences in understanding proposed in Skemp’s model of intelligent learning are clarified through his use of a metaphor for mental schemas. Figure 1 is an example of the kind of network Skemp introduced as a metaphor for the understanding of a mental schema.



Figure 1. Skemp’s schema metaphor.

It should be remembered that such diagrams are not intended to be in any sense a physical representation of the structure of the brain or the way in which it stores data, but are simply a metaphor to assist our perception of the cognitive structures of the mind and our discussion of the storage and manipulation of concepts affecting our thinking, learning and understanding. The qualitative differences in understanding of Skemp's model of intelligent learning are represented in this diagrammatic metaphor through references to associative, or A-links, between concepts where one has instrumental understanding, and conceptual, or C-links, for relational understanding. To justify the idea of concept links Skemp argues (1979c, p. 131) that since "activation of one concept can activate, or lower the threshold for, others." then there must be connections between them, and he adds:

This idea has been developed to include a dimension of strong or weak connections, whereby activation of a particular concept results in the activation of others within quite a large neighbourhood, or within only a small one. Recently I have become interested in another difference which is not quantitative as above, but qualitative. This is between two kinds of connection which I call *associative* and *conceptual*: for short A-links and C-links.

He illustrates the difference between the two types of connections with number sequences. The numbers 2, 5, 7, 0, 6 are concepts which are connected by A-links; there is "no regularity which can give a foothold for the activity of intelligence." (*ibid*, p. 187). In contrast the numbers 2, 5, 8, 11, 14, 17 have a conceptual connection – a common difference of 3 – and so have a common C-link. Clearly, a link may be either associative or conceptual for any given person depending on whether they have formed the connecting concept. I have sometimes illustrated this by asking students whether they could remember, for a week, the following sequence:

7, 8, 5, 5, 3, 4, 4, 6, 9, 7, 8, 8

Most immediately decide, on the basis of A-links, that they could not, but once they are given the concept that these are the number of letters in the months of the year, the C-link is formed and they agree that they could. This demonstrates the principle that A-links can change to C-links, through the use of reflective intelligence, and such changes are accompanied by reconstruction of the appropriate schemas. One of Skemp's endearing teaching points is his use of personal experience and

I have identified in my own experience, something exemplifying this transition. Like many of us, no doubt, I have committed to memory, using A-links, certain results in mathematics. One in particular which I had trouble remembering was from the trigonometric results, namely, which of $\tan^{-1}(\sqrt{3})$ and $\tan^{-1}(1/\sqrt{3})$ was 30° and which 60° . My A-link was in a state of constant degeneration. I can remember now my feelings of stupidity when one day I formed the C-link that since $(1/\sqrt{3}) < 1 < \sqrt{3}$, then $\tan^{-1}(1/\sqrt{3}) < \tan^{-1}(1) < \tan^{-1}(\sqrt{3})$, and since $\tan^{-1}(45^\circ) = 1$ was already a C-link for me, from a 45° isosceles triangle, the rest followed instantly. It seems to me that consciously capturing such moments of insight brought about by reflective intelligence is quite rare, but especially valuable.

As the above example illustrates, a disadvantage of A-links is that they have to be memorised by rote, whereas for a C-link “we can put a name to it, communicate it, and make it an object for reflective intelligence: with all the possibilities which this opens up.” (*ibid*, p. 188). Unsurprisingly, the quality of our schemas is a key determinant of our success in mathematics and Skemp (*ibid*, p. 189) observes the importance of C-links in this:

But the larger the proportion of C-links to A-links in a schema, the better it is in several closely related ways. It is easier to remember, since there is only one connection to learn instead of many. Extrapolation is often possible and even inviting, as some will find in the cases of sequences of C-linked numbers. And the schema has an extra set of points at which assimilation, understanding, and thus growth can take place.

Thus, according to Skemp’s theory of intelligent learning, a key objective of mathematics teaching should be to provide experiences which encourage and promote the formation of knowledge structures which, wherever possible, comprise conceptual links (C-links) corresponding to relational learning. It is exactly these types of knowledge structures which are necessary to promote *versatile* thinking since it is only through conceptual links that the global and sequential aspects of mathematics, and the different representations of concepts, can be properly related. What may not be so clear is what activities in the mathematics classroom we can use to encourage the formation of such schemas.

Developing Versatility of Mathematical Thought

The relationship between processes (which I shall try to use as a generic term, reserving procedure for a specific algorithm for a given process) and objects has been the subject of much scrutiny by a number of researchers in recent years. Skemp's (1961 p. 47) insight that:

For the algebraist will continue, in due course, to develop concepts of new classes of numbers (e.g., complex numbers) and new functions (e.g., gamma functions) by generalising the field of application of certain operations (taking square roots, taking the factorial of a number); and will study the application of the existing set of operations to the new concepts.

describes the construction of new mathematical objects (concepts) by generalising operations. Davis (1984, pp. 29–30) formulated a similar idea:

When a procedure is first being learned, one experiences it almost one step at a time; the overall patterns and continuity and flow of the entire activity are not perceived. But as the procedure is practised, the procedure itself becomes an entity – it becomes a *thing*. It, itself, is an input or object of scrutiny. . . . The procedure, formerly only a thing to be done – a verb – has now become an object of scrutiny and analysis; it is now, in this sense, a noun.

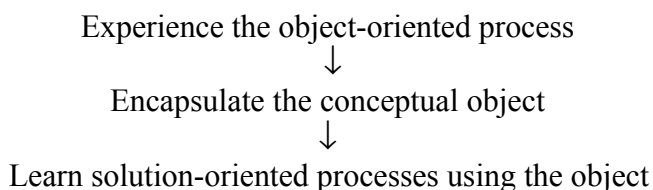
Describing here how a procedure becomes an object he strikes at a key distinction between the two when he mentions the 'one step at a time' nature of procedures when they are first encountered, in contrast with 'the overall. . . flow of the entire activity', or the holistic, object-like nature which they can attain for an individual. More recently others have spoken of how an individual *encapsulates* or *reifies* the process so that it becomes for them an object which can be symbolised as a *procept* (Dubinsky & Lewin, 1986; Dubinsky, 1991, Cottrill *et al.*, 1997; Sfard, 1991, 1994; Gray & Tall, 1991, 1994). However, it seems that there are at least two qualitatively different types of processes in mathematics; those *object-oriented* processes from which mathematical objects are encapsulated (see Tall *et al.* in press) and those *solution-oriented* processes which are essentially algorithms directed at solving 'standard' mathematical problems. An example of the first would be the addition of terms to find the partial sums of a series, which leads to the conceptual object of limit and the second could be exemplified by procedures to solve linear algebraic equations. Much mathematics

teaching in schools has concentrated on *solution-oriented* processes to the exclusion of *object-oriented* processes. However *solution-oriented* processes usually operate on the very objects which arise from the *object-oriented* processes. Hence ignoring these is short sighted and will prove counter-productive in the long term. Failure to give students the opportunity to encapsulate *object-oriented* processes as objects may lead them to engage in something similar to the *pseudo-conceptual* thinking described by Vinner (1997, p. 100) as exhibiting “behaviour which might look like conceptual behaviour, but which in fact is produced by mental processes which do not characterize conceptual behaviour.” In this case students may appear to have encapsulated processes as objects but this turns out on closer inspection to be what we will call a *pseudo-encapsulation*. For example, in arithmetic students may appear to be able to work with a fraction such as $\frac{4}{5}$ as if they have encapsulated the process of division of integers as fractions. However, many have not done so. I am still intrigued by the first time I encountered such a perception. It was in 1986 (see Thomas, 1988; Tall & Thomas, 1991) when I discovered that 47% of a sample of 13 year-old students thought that $6 \div 2x + 17$ and $\frac{6}{7}$ were not the same, because, according to them, the first was a ‘sum’ but the second was a ‘fraction’. Such students are at a stage where they think primarily in terms of *solution-oriented* processes, and in this case they had not encapsulated division of integers as fractions. Instead they had constructed a *pseudo-encapsulation* of fraction as an object, but not one arising from the encapsulation of division of integers.

Similarly in early algebra many students may work with symbolic literals in simplifying expressions like $3a + 2b - 2a + b$ as if they view them as procepts. However, rather than having encapsulated the process of variation as a procept, symbolised by a or b , they are actually thinking of them as *pseudo-encapsulations*, concrete objects similar to ‘apples’ and ‘bananas’. When dealing with expressions such as $2x + 1$, a common perspective enables students to work with it but see it as a single arithmetic result not as a structural object representing the generalisation of adding one to two times any number x . When they move on to equations these students can use their *pseudo-encapsulations* to solve equations such as $2x + 1 = 7$. They may appear to be using algebraic methods to get their answer but they are actually

thinking arithmetically, and given the answer 7, either use trial and error, or work backwards numerically, to find x . This becomes clear when they are unable to solve equations similar to $2x + 1 = 7x - 3$ (Herscovics & Linchevski, 1994).

It is possible that students might be assisted to avoid engaging in the construction of *pseudo-encapsulations* by approaching the learning of mathematics (in different representations) in the following way.



This may be illustrated by the learning of the derivative of a function, $f'(x)$ or $\frac{dy}{dx}$. Students should first experience the limiting process of $\frac{f(x+h)-f(x)}{h}$ (or equivalent) tending to a limit as h tends to 0 in the symbolic representation, and the gradient of a chord tending to the limit of the gradient of a tangent in the graphical representation. Then they will be able to encapsulate the derived function as an object encapsulated from this process. This will mean that they are able to operate on it, enabling them to understand the meaning of $\frac{d(\frac{dy}{dx})}{dx}$ in terms of rates of change, which otherwise will prove very difficult. Many students, who learn the derivative as a solution-oriented process, or algorithmic rule, are in precisely the position of having no meaning for the second derivative and are left having to apply it as a repeated process, probably devoid of any other meaning in any representation.

The *versatile* approach to the learning of mathematics which is being espoused here recognises the importance of each of the three stages of the recurring trilogy above and stresses the importance of experiencing each step in as many different representations as possible in order to promote the formation of conceptual or C-links across those representations. Students need personal experience of object-oriented processes so that they can encapsulate the objects. The *versatile*

learner gains the ability to think in a number of representations both holistically about a concept or object, and sequentially about the process from which it has been encapsulated. A major advantage of a *versatile* perspective is that, through encapsulation, one may attain a global view of a concept, be able to break it down into components, or constituent processes, and conceptually relate these to the whole, across representations, as required. Without this one often sees only the part in the context of limited, often procedural, understanding and in a single representation.

How though does one encourage versatile thinking? What experiences should students have so that they may build such thinking and conceptual links? We have seen that reflective intelligence is a key to the transition from process to object, and hence to progress in mathematics (although this is not the only way objects may be formed in mathematics – see Davis, Tall & Thomas, 1997; Tall *et al.*, 1999) and my contention is that representation-rich environments which allow investigation of the object-oriented processes which give rise to mathematical objects will encourage reflective thinking.

Skemp (1971, p. 32) appreciated that “Concepts of a higher order than those which a person already has cannot be communicated to him by a definition, but only by arranging for him to encounter a suitable collection of examples.” Whilst this is a valuable principle, it somewhat oversimplifies the situation. Many higher order concepts are not abstracted from examples but, as we have discussed above, have to be encapsulated by the individual from object-oriented processes (see Tall *et al.*, 1999). Hence the examples which the student needs are not simply those embodying the finished concept itself, but also the processes which give rise to it.

A Newton-Raphson Example

Many of the points made in the above discussion can be well exemplified by the Newton-Raphson method for calculating the zeroes of a differentiable function. Given the formula:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)},$$

students may have no difficulty working in a solution-oriented manner to calculate the zeroes of a function, such as a polynomial, to any required accuracy. However, asking them questions about what it is

they are doing, and why it works, is likely to elicit blank stares. They might reason that *doing* the mathematics and getting the right answer is the main thing, so why would one want to know anything further?

Furthermore, it is of interest that the formula above, in the usual form it is given, is that best suited to a procedural calculation of x_2 , given x_1 . To emphasise understanding rather than convenience of calculation a better form, which follows directly from a graphical representation, such as Figure 1 where x_2 can be *seen* to be a better approximation to the zero than x_1 , would be:

$$f'(x_1) = \frac{f(x_1)}{x_1 - x_2}.$$

This can then easily be rendered in the usual procedural form for calculations. This version equates two symbolic representations of the gradient of a tangent, which are easily linked via the graphical format. The *versatile* approach in this case is to look at the object-oriented or concept-oriented processes first, especially in a visually rich graphical representation, *before* a consideration of the *solution-oriented* process. This can be accomplished by allowing students to experience graphs of functions, such as sine and cubic curves (for example using a graphic calculator or suitable computer graphing package), and to construct meaning for the process of ‘zooming in’ on a zero by drawing successive tangents at points on the curve. A few minutes experience with diagrams such as that in Figures 2 and 3 can provide tremendous insight into when and why this method works, and where the formula comes from.

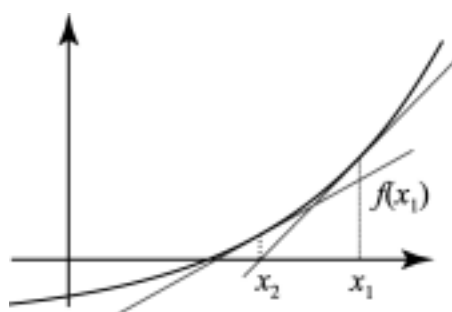


Figure 2. A diagram illustrating a successful search for a function's zeroes.

In this way students can reflect on what the process is doing, where the formula for finding a zero comes from, and most importantly under what conditions it may succeed or fail, emphasising, for example, the importance of the first approximation.

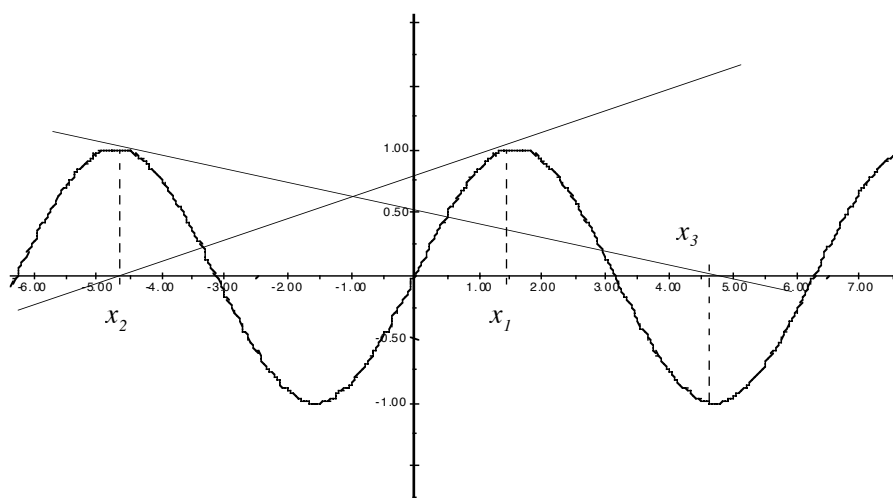


Figure 3. An example of what can go wrong with the Newton-Raphson method.

Thus they may visually conceptualise the process of finding the zeroes. Once this is established the students can begin to learn how to calculate values of zeroes of functions using a symbolically based solution-oriented process.

Other Examples

In arithmetic children become accustomed to working in an environment where they solve problems by producing a specific numerical “answer” and this leads to the expectation that the same will be true for algebra. It isn’t. Such pupils face a number of conceptual obstacles to their progress in algebra (see e.g. Tall & Thomas, 1991; Linchevski & Herscovics, 1996). The concept of variable forms one major barrier between arithmetic and algebra, and yet it is a concept not often explicitly addressed in teaching which focuses on solution-oriented processes. Skemp (1971, p. 227) expressed the same point, commenting that, “The idea of a variable is in fact a key concept in algebra – although many elementary texts do not explain or even mention it.” A versatile approach to the learning of variables seeks to build experience of the mathematical process of variation in a manner which blends personal investigation with the incorporation of visual representations. In a study to investigate this (Graham and Thomas, 1997, 2000) a graphic calculator was used to help students to a versatile view of variable, as both a varying process and an object symbolised by a letter. The 13 year-old children were encouraged to

‘see’ the calculator as storing numbers in its lettered stores, with each store comprising a location for the value and a name, a letter. A major advantage of the graphic calculator in this work is that it preserves several of these operations and their results in the view of the student, prolonging their perception time, and the opportunity to reflect on what they have done and hence form C-links. A feature of the research which encouraged such reflective thinking was the use of *screensnaps*, such as that in Figure 4. Here one is required to reproduce the given screen on the graphic calculator. This is a valuable exercise which students attempt, not by using algebraic procedures, but by assigning values to the variables and predicting and testing outcomes.

A+B	11
A-B	-3

Figure 4: Example of a *screensnap* from graphic calculator algebra research.

Exercises such as this have the advantage of encouraging beginning algebra students to engage in reflective thinking using variables, requiring them to: reflect on their mental model of variable; ‘see’ the variable’s value as a changeable number; physically change the value of a variable; and relate the values of two variables.

These experiences significantly improved understanding of the way students understood the use of letters as specific unknown and generalised number in algebra. For example, when asked a question similar to that used by Küchemann (1981) in his research, namely:

Does $x - y = z - y$ — always, never or sometimes ... when?,

30% of a graphic calculator group of 130 answered ‘sometimes when $x = z$ ’, compared with 16% of the 129 controls ($\chi^2 = 7.73$, $p < 0.01$), who had learned algebra in a traditional, procedural, solution-oriented way. Correspondingly, one of the students who had used the graphic calculators was asked on interview:

Int: Did you have any previous knowledge of algebra?

N: Yeah, we did some in form 2, a little bit, but I didn’t really understand much of it.

... Int: $L+M+N=L+P+N$. Now what would be the answer there do you think? [ie Are they equal always, never or sometimes ... when?]

N: Sometimes, if M and P are the same value.

Int: OK good. What about the next one, $A+B=B$?

N: Um, sometimes if $A=0$.

Such questions cannot easily be answered by students who, having constructed a *pseudo-encapsulation*, see letters as concrete objects, since they will not appreciate that two different objects can be the same. Rather they require one to understand that the letters represent values, which may coincide. N had seen this, was happy to let different letters have the same value and could pick out the correct value 0 for A in the second example. It was apparent that the model had assisted in the attainment of this view, since one student remarked:

I think the STORE button really helped, when we stored the numbers in the calculator. I think it helped and made me understand how to do it and the way the screen showed all the numbers coming up I found it much easier than all the other calculators which don't even show the numbers.

The power of a representation which promotes visualisation is seen here. The way the calculator screen preserves several computations on the variables has assisted this student to build ideas about the use of letters in algebra to represent numbers. These results and comments demonstrate that these students had not only gained the flexibility of insight where they could see two letters as representing a range of numbers which could sometimes coincide, but, as their simplification results showed, they could operate on these encapsulated objects at least as well as students who had spent their time on manipulation techniques. They had a more *versatile* conception of variable.

Many students carry forward a procedural, arithmetic perspective into their acquaintance with expressions and equations in algebra. Substitution of a value into an expression or function is not a problem for such students because they have an arithmetic procedure they can follow. However, many students have not encapsulated expression as a variable-based object and so when faced with simplification of expressions or solving equations they build meaning based on concrete objects and then engage in solution-oriented processes. This view of equation is a real problem when they are required to solve linear algebraic equations of the form $ax + b = cx + d$. For the student who is thinking sequentially and looking for the result of a calculation, such a problem is telling them that two procedures give the same result, but

they are not told what that result is. Thus, unable to work either backwards from the answer, or forwards to a known answer, many fall at this obstacle, which has been called the *didactic cut* (Fillooy and Rojano, 1984) or *cognitive gap* (Herscovics and Linchevski, 1994).

A *versatile* view of equation can be promoted by a computer environment such as that of the *Dynamic Algebra* program (see Figure 5), which encourages students to construct equations in terms of variable and expression objects which can be simultaneously evaluated (Thomas & Hall, 1998).

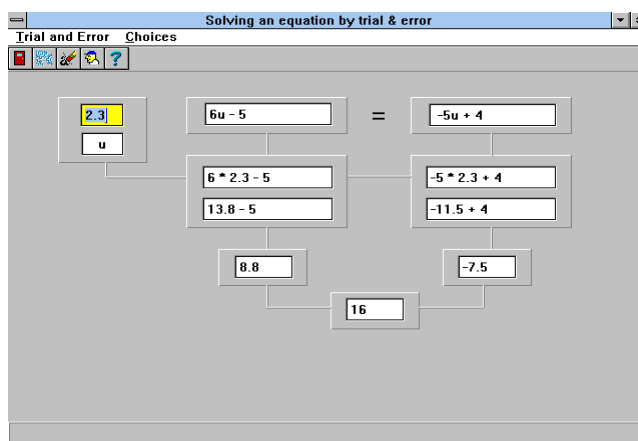


Figure 5. A screen from the *Dynamic Algebra* program.

The representation employed in this approach emphasises the visual aspects of variable mentioned above (as a location and label) and equation (as two equal expression boxes) in an environment where a number of processes for equation solving, including trial and error substitution and balancing can be investigated. The trial and error experiences combine the visual model of variable with the idea of substitution in an expression (or function) and equivalence of expression (or function). Students investigate solutions by entering values for the variable, here u , until both sides are seen to be equal in value, that is the difference between the two sides is zero. This reinforces in the student's experience, the results of evaluating the expressions constituting each side of the equation. In this way students can conceptualise equation before embarking on solution-oriented processes. One student, who used guess and substitution to solve linear equations before using this program, when interviewed about

questions with more than one n , such as $2n = n + 6$, showed little conceptual appreciation of equation asking, “Does n have a variety of numbers or are they the same?” After investigating equations using the computer environment she appeared to have no problem with this and her solution to the equation $5n + 12 = 3n + 24$ was written as:

$$\begin{aligned}
 5n - 3n + 12 &= \cancel{3n} - \cancel{3n} + 24 \\
 2n + 12 &= 24 \\
 2n + \cancel{12} - \cancel{12} &= 24 - 12 \\
 2n &= 12 \\
 n &= 6
 \end{aligned}$$

The computer program was structured by this procedure of performing the same operation on each side of the equation, although the students were not taught it explicitly but had to build up their own understanding. This student had progressed through an investigation of object-oriented processes to an encapsulation of equation as a statement of equality of expressions and hence was able to operate on it using a solution-based procedure. To improve in ability to the point where she could solve such equations after a few hours reflective activity is no small achievement, and indicates the possible value of a versatile approach to the learning of expression and equation.

While introductory algebra finds its place at one end of the secondary school curriculum, the definite (or Riemann) integral is firmly at the other. However, here too, it is not uncommon to find students who are instrumentally engaged in following set solution-oriented procedures for calculating integrals with little idea of what the integrals are or represent. They are *process-oriented* (Thomas, 1994), locked into a mode of calculating answers using antiderivatives without the directing influence of a holistic, structural understanding. The inability of such students to deal with questions which do not appear to have an explicit solution procedure was highlighted in another study (Hong & Thomas, 1997, 1998). For example, a generalised function notation proved particularly difficult for many of these students to cope with, since it did not fit into their process-oriented framework. In the question:

If $\int_1^3 f(t)dt = 8.6$, write down the value of $\int_2^4 f(t-1)dt$,

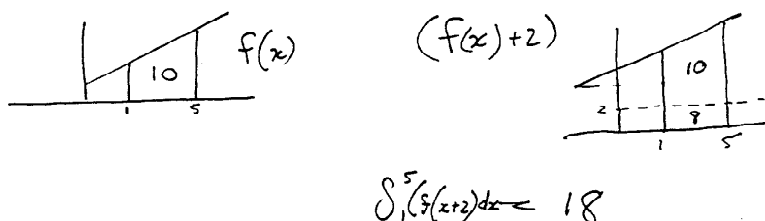
a number of students who were unable to see how to apply a known procedure to deal with $f(t-1)$, used a number of different procedural ways to surmount the obstacle including taking f as a constant, using $f(t-1)$ or attempting to integrate the $t-1$ and obtaining $t^2 - t$ or $\frac{(t-1)^2}{2}$.

The students were introduced to visually rich computer environments, using Maple and Excel which combined symbolic, tabular and graphical representations. Students could calculate upper and lower area sums (in more than one representation), find the limits of these sums, relate the sign of the definite integral to the area under the graph of the function, consider the effect of transformations on areas, and begin to think of area as a function. In this way these computer experiences allowed a personal cross-representation investigation of the processes lying behind these concepts of integration, assisting the students to construct conceptual objects associated with Riemann sums and integrals.

Having investigated the area under the functions x^2 , $x^2 + 2$ and $(x - 2)^2$ they were able to extend their understanding to cope with the question,

If $\int_1^5 f(x)dx = 10$, then write down the value of $\int_1^5 (f(x) + 2)dx$,

which considered a transformation parallel to the y -axis for a general function $f(x)$. 44% of the school students were able to deal with integrating $f(x) + 2$ after their computer experiences, compared with only 11% before. A student unable to make any attempt at the pre-test, drew a post-test diagram showing clearly that he understood this question visually in terms of the creation of an extra area by the transformation parallel to the y -axis.

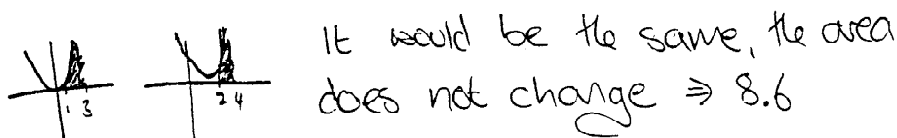


When the transformation (see the question above) was parallel to the x -axis, and involved a general function $f(t-1)$, the computer group's performance improved from 11%, to a success rate of 50%. One student demonstrated that she was now thinking in terms of conservation of area and was able to relate this to the procepts by writing:

$$\int_1^3 f(t) dt = \int_2^4 f(t-1) dt. \therefore \text{Area the same.}$$

$$\therefore \int_2^4 f(t-1) dt = 8.6$$

her reason for the equality being '[because] Area the same'. Another student who gave no response at the pre-test showed her improved conception that the translation of the graph leaves the area unchanged when in the post-test she was able to visualise that after the transformation parallel to the x -axis the area would not be changed. This understanding was manifest through her working, when she drew the following diagrams as part of her solution:



These examples show the value of the linking of graphical, visual experiences with other representations, and their effect on the form of some of the students' reasoning and answers. In contrast other students were able to symbolise the relationship very precisely in terms of procepts by writing $\int_2^4 f(t-1) dt = \int_1^3 f(y) dy$, and hence give the solution as 8.6. There was evidence of a versatility in approach from the students who had worked on the computer.

The Role of Visual Representation: Extending Skemp's Model

The research projects and ideas outlined above relied heavily on the use of representations of concepts using visual imagery; for the graphs of functions, for a variable, as a store with a value and a label, for equations, and for the way in which areas relate to integration. The importance of visual representations to encapsulation of conceptual objects is one which I believe is at the core of learning and one which

intrinsically involves the formation of conceptual structures, with their C-links and A-links. Secondly, I believe that imagery plays a far more important part in all our thinking than we are, literally, aware of. Much of our thinking (for example the processing of the majority of data from our visual field) appears to be unconscious, as Johnson-Laird (1993, p. 354) comments "... there are also many benign unconscious processes that underlie perception and cognition."

The majority of such unconscious processing appears to be holistic in nature and extremely fast, compared with the generally sequential and slower processing of the conscious mind (Gazzaniga, 1985). Imaginal input, is unlikely to be randomly stored in the brain but, along with other holistic data, such as aural input, for example voices and music, is also formed into schemas, similar to those of our logical and verbal thinking.

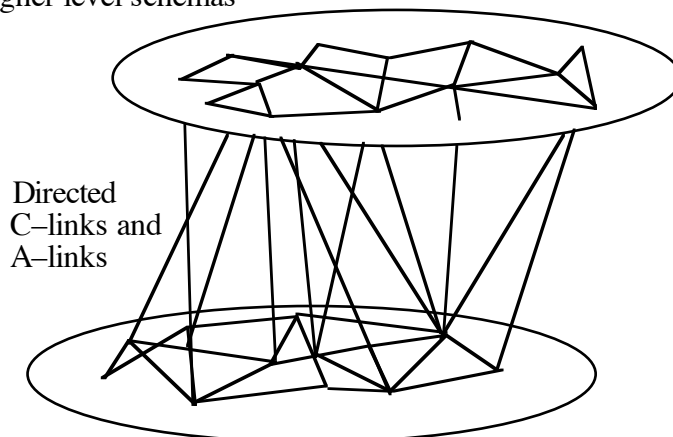
I have designated the consciously accessible schemas as higher level thinking, and the unconscious as lower level thinking, more to emphasise their qualitatively different natures than to elevate one above the other. Rather than being formed into two totally separate and distinct schematic structures there is much circumstantial evidence that the two are linked.

I believe that there is a *cognitive integration* (Thomas, 1988) of the schemas at these two levels of thinking with both C-links and A-links between concepts etc. at the two levels which can be encouraged and enhanced. For example one may see a face in a crowd or hear a piece of music which one appears to recognise but is unable to relate to in any other way, such as by naming. This may signify that an image of the face or music pattern in the lower level schemas is linked associatively (A-link) to those at the higher level which mediate recognition. In contrast one may see the face and exclaim "Hi, uncle John" or hear the music and comment that, "I prefer von Karajan's version of this Beethoven symphony." Such remarks are evidence of conceptual links (C-links) between the schemas at the lower and higher levels. Figure 6, building on Skemp's diagram, is a metaphor for the relationship between the two qualitatively different modes of thinking.

The 'vertical' links between the two levels allow the mental imagery schemas to influence the higher level cognitive functions of the mind. This approach leads to some valuable insights about why we

find some intellectual goals hard to achieve and yet others, which we may not wish, seem to thrust themselves upon our consciousness.

Higher level schemas



Lower level schemas

Figure 6. A simplified model for Cognitive Integration.

How may this be explained? In the mental process of finding a path from a present state P, to a goal state G, there will be at virtually every stage of the path, different possibilities for the route, many (if not all) of which will involve unconscious processing. It is likely that much of the unconscious processing will occur simultaneously and across representations, where the C-link connections exist.

At any nodal point, the path most likely to be taken is that which has the highest strength factor. Thus these tend to dominate our thinking process at each stage. This is not to say that we must take this particular route, but simply that it is more prominent and thus more likely to be taken.

To illustrate, if we travel from one place to another in a car and wish to get to our destination in the most efficient way we may likely travel along the motorway, if there is one. Periodically we will approach junctions, with roads of varying size leaving the motorway, ranging from small country roads to other motorways. Given a choice, and the desire for the most efficient journey, we will probably choose to use the major route at each junction. We do however have a choice, and other factors may influence our decision, such as our level of tiredness, boredom, interest in a passing feature of the landscape, likely traffic jams etc.

With intelligent learning, just as with this illustration, it is often the case that, in order to follow the most efficient route, indeed sometimes in order to gain access to any major route at all, we must first avail ourselves of minor routes. In the context of our thought processes, to make a positive decision to choose a path of lower strength factor rather than one of a higher factor requires reflective intelligence, since it is this which makes us aware of the choices. A mathematical example which illustrates the problem students may have with finding what for them is often a minor cognitive pathway is given by this question:

Solve the equation $(2x + 3)^2 = 9$.

Many students immediately link this question to quadratic equations (which it is of course), but these in turn are often very strongly linked to the 'standard' form

$$ax^2 + bx + c = 0$$

which is in turn strongly linked to the 'formula', a solution-oriented procedure for solving such an equation. Hence they may be pushed along the procedural route of multiplying out the brackets, collecting like terms, simplifying and then using this formula. In doing so they miss the 'minor' cognitive path which, via the step

$$2x + 3 = \pm 3,$$

leads much more directly to the goal state of a solution to the equation. Finding this minor path can be facilitated by versatile thinking. A global view, promoted by suitable imagery, can enable one to see the structure $A^2 = B^2$ in the question, leading to $A = \pm B$.

How may students be encouraged to think in this versatile way? One necessary initiative is for teachers to copy the example of the esoteric, boutique tour guide who sees it as his job not only to make sure that the coach party is taken down the small country roads but that they get to stop off at all the places of interest on the way, take photographs (images) as reminders, observe the structural form of buildings etc. and receive a commentary on why they are of interest. Sadly many teachers who would love to teach in this versatile way on their mathematical 'tour', visiting a number of available representations of each concept they introduce, do not have the luxury of the time to do so. They are constrained by the 'directors' of the tour company

who insist that schedules are strictly kept to and standards and deadlines are met.

How then does the concept of *cognitive integration* help us to address the vital question of what may be done to help the solution process-oriented student gain a versatile conception of algebra and other branches of mathematics? In view of the above model, two possible measures, which are not mutually exclusive, have been stressed here. First we may argue that students are best helped to a versatile view of mathematics by an approach which emphasises experience and use of processes leading to mathematical constructs before being presented with finished mathematical objects and corresponding solution processes. The second conclusion is that more emphasis should be given to the global/holistic view of the mathematics by promoting different representations and the links between them at all stages. As educators, the more we incorporate these into our mathematics teaching the more we make available links, which are often visual, to assist understanding and learning, with the possibility that versatile thinking may result. However, one should be conscious to introduce the representations in a way which encourages conceptual or C-links between the symbolic, the procedural and the visual, as in the Newton-Raphson example we started with. Such conceptually linked imagery will increase the possibility of a student being able to vary their cognitive focus in a given mathematical situation from a sequential, left-to-right process-led perspective to a global/holistic concept-driven mode, or vice-versa.

These ideas on versatile learning have been firmly based on the pioneering insights of Richard Skemp. The quality of his thinking and the presentation of his ideas continue to move mathematics education forward even today. The day that Richard died, I later found out, I had spent discussing mathematics education with David Tall, and somehow that seemed very appropriate.

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